Circular Electric Wave Transmission in a Dielectric-Coated Waveguide

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The mode conversion from the TE_{01} *wave to the* TM_{11} *wave which normally occurs in a curved round waveguide may be reduced by applying a uniform dielectric coat a few mils thick to the inner wall of the waveguide. Such a dielectric coat changes the phase constant of the* TM_{11} *wave without affecting appreciably the phase and attenuation constants of the* TE_{01} *wave. Thus, relatively sharp bends can be negotiated to change the direction of the line or deviations from straightness can be tolerated. For each of these two cases a different optimum thickness of the dielectric coat keeps the* TE_{01} *loss at a minimum. At 5.4-mm wavelength a bending radius of 8 ft in a* $\frac{7}{8}$*-inch pipe or 50 ft in a 2-inch pipe can be introduced with 0.2 db mode conversion loss. Deviations from straightness corresponding to an average bending radius of 300 ft can be tolerated in a 2-inch pipe at 5.4-mm wavelength with an increase in* TE_{01} *attenuation of 5 per cent. Serpentine bends caused by equally spaced supports of the pipe may, however, increase the mode conversion and cause appreciable loss at certain discrete frequencies. Compared with the plain waveguide the dielectric-coated guide behaves more critically in such serpentine bends. The mode conversion from* TE_{01} *to the* TE_{02}, TE_{03}, $\cdots$ *waves at transitions from a plain waveguide to the dielectric-coated guide is usually very small.*

I. INTRODUCTION

In a curved section of cylindrical waveguide the circular electric wave couples to the TE_{11}, TE_{12}, TE_{13} $\cdots$ modes and to the TM_{11} mode.[1] The coupling to the TM_{11} mode presents the most serious problem since the TE_{01} and TM_{11} modes are degenerate, in that they have equal phase velocities in a perfectly conducting straight guide. In a bend all TE_{01} power introduced at the beginning will be converted to the TM_{11} power at odd multiples of a certain critical bending angle. One can reduce this complete power transfer by removing the degeneracy between TE_{01} and

TM_{11} modes. The finite conductivity of the walls introduces a slight difference in the propagation constants and in a 2-inch pipe at 5.4 mm wavelength a bending radius of a few miles can be tolerated with about double the attenuation constant of the TE_{01} wave. To get more difference in the phase constants of the TM_{11} and TE_{01} modes, one might consider a dielectric layer next to the wall of the waveguide. Since the electric field intensity of the TE_{01} mode goes to zero at the wall but the electric field intensity of the TM_{11} mode has a large value there, one might expect a larger effect of the dielectric layer on the propagation characteristics of the TM_{11} wave than on the TE_{01} wave.

In doing this, however, one has to be aware of the influence the dielectric layer will have on the propagation characteristics of the TE_{1m} modes which also couple to the TE_{01} wave in curved sections. The TE_{12} wave couples most strongly to the TE_{01} wave and of the TE_{1m} family it is the next above the TE_{01} wave in phase velocity. With the dielectric layer one has to expect this difference in phase velocity to be decreased and consequently the mode conversion to the TE_{12} wave to be increased.

In the next section we will solve the characteristic equation of the cylindrical waveguide with a dielectric layer for the normal modes and arrive at approximate formulas for the phase constants. We will also compute the increase in attenuation of the normal modes as caused by the dielectric losses in the layer. The change in wall current losses as caused by the dielectric layer is of importance here only for the TE_{01} wave and we will calculate it only for this wave.

In order to know what bending radii can be tolerated with a dielectric coat, we have to evaluate the coupled wave theory[2] for small differences in propagation constants between the coupled waves. This will be done in Section III. Especially we will consider serpentine bends which occur in any practical line with discrete supports. In that situation, at certain critical frequencies, when the supporting distance is a multiple of the beat wavelength between TE_{01} and a particular coupled wave, we will have to expect serious effects on the propagation constant of the TE_{01} wave.

In Section IV we will combine the results of Sections II and III and establish formulas and curves for designing circular electric waveguides with a dielectric coat. We will distinguish there between two different applications. The first case is to negotiate uniform bends of as small a radius as possible and the second is to tolerate large bending radii which may occur in a normally straight line.

At transitions from plain waveguide to the dielectric-coated guide, power of an incident TE_{01} wave will be partly converted into higher

TE_{om} waves. In Section V we shall calculate the power level in the TE_{om} waves resulting from this conversion.

II. THE NORMAL MODES OF THE DIELECTRIC-COATED WAVEGUIDE

The waveguide structure under consideration is shown in Fig. 1. To find the various normal modes existing in this structure, Maxwell's equations have to be solved in circular cylindrical coordinates in the air-filled region 1 and dielectric-filled region 2. The boundary conditions are: equal tangential components of the electric and magnetic field intensities at the boundary ($r = b$) between regions 1 and 2 and, assuming infinite conductivity of the walls, zero tangential component of the electric field at the walls ($r = a$). Upon introducing the general solutions into these boundary conditions, we get a homogeneous system of four linear equations in the amplitude factors. Non-trivial solutions of this system require the coefficient determinant to be zero. This condition is called the characteristic equation. Solutions of the characteristic equation represent the propagation constants of the various modes. These calculations have been carried out elsewhere,[3, 4] and the characteristic equation arrived at there has the following form:

$$n^2 \left[\frac{1}{x_1^2} - \frac{1}{x_2^2}\right]^2 - \rho^2 \frac{x_2^2 - x_1^2}{x_2^2 - \epsilon x_1^2} \left[\frac{1}{x_1} \frac{J_n'(\rho x_1)}{J_n(\rho x_1)} + \frac{\epsilon}{\rho x_2^2} \frac{W_n(x_2, \rho x_2)}{U_n(x_2, \rho x_2)}\right] \cdot \left[\frac{1}{x_1} \frac{J_n'(\rho x_1)}{J_n(\rho x_1)} + \frac{1}{\rho x_2^2} \frac{V_n(x_2, \rho x_2)}{Z_n(x_2, \rho x_2)}\right] = 0. \tag{1}$$

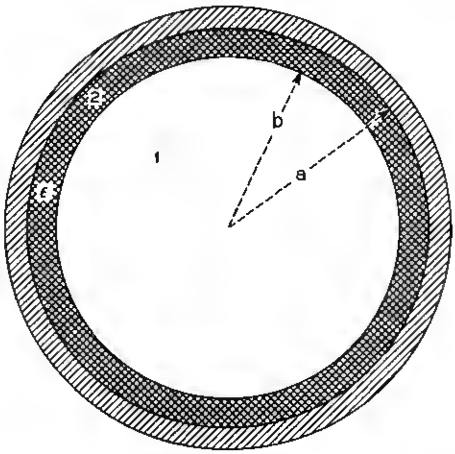

FIG. 1 — The dielectric-coated waveguide; $\rho = b/a$, $\delta = 1 - \rho = (a - b)/a$.

Here we have used the following definitions:

$$
\begin{aligned}
U_n(x, \rho x) &= J_n(\rho x)N_n(x) - N_n(\rho x)J_n(x), \\
V_n(x, \rho x) &= \rho x^2[J_n'(\rho x)N_n'(x) - N_n'(\rho x)J_n'(x)], \\
W_n(x, \rho x) &= \rho x[J_n(x)N_n'(\rho x) - J_n'(\rho x)N_n(x)], \\
Z_n(x, \rho x) &= x[J_n'(x)N_n(\rho x) - J_n(\rho x)N_n'(x)].
\end{aligned}
\tag{2}
$$

J_n and N_n are cylinder functions of the first and second kind, respectively, and the prime marks differentiation with respect to the argument. The other symbols are:

$$
\begin{aligned}
\rho &= b/a \text{ ratio of radii}, \\
x_1 &= \xi_1 a\,, \\
x_2 &= \xi_2 a\,.
\end{aligned}
\tag{3}
$$

The relative dielectric constant of region 2 is indicated by ϵ which is assumed to be complex to take dielectric losses into account. The radial propagation constants ξ_1 and ξ_2 are related to the axial propagation constant γ and the free-space propagation constant $k = 2\pi/\lambda$ by

$$
\begin{aligned}
\xi_1^2 &= k^2 + \gamma^2, \\
\xi_2^2 &= \epsilon k^2 + \gamma^2.
\end{aligned}
\tag{4}
$$

The circumferential order of the solution is indicated by n.

For $n = 0$, equation (1) splits into the two equations

$$\frac{1}{x_1}\frac{J_0'(\rho x_1)}{J_0(\rho x_1)} + \frac{\epsilon}{\rho x_2^2}\frac{W_0(x_2\,, \rho x_2)}{U_0(x_2\,, \rho x_2)} = 0, \tag{5}$$

and

$$\frac{1}{x_1}\frac{J_0'(\rho x_1)}{J_0(\rho x_1)} + \frac{1}{\rho x_2^2}\frac{V_0(x_2\,, \rho x_2)}{Z_0(x_2\,, \rho x_2)} = 0 \tag{6}$$

representing the TM_{om} and TE_{om} waves, respectively.

Except for $n = 0$ the solutions of (1) and the modes in tbe waveguide do not have transverse character as in the case of a uniformly filled waveguide. They are hybrid modes. However, it is reasonable to label them as TE_{nm} or TM_{nm}, according to the limit which they approach as the dielectric layer becomes very thin.[5]

Modes in the dielectric-coated guide with a very thin coat may be treated as perturbed TE_{nm} or TM_{nm} modes of the ideal circular waveguide. The perturbation terms are found by expanding (1) for small

values of $\delta = 1 - \rho$. This is done in Appendix 1. The perturbation of the propagation constants for the various normal modes is:

$$\mathrm{TM}_{nm} \quad \frac{\Delta\gamma}{\gamma_{nm}} = \frac{\epsilon - 1}{\epsilon}\,\delta, \tag{7}$$

$$\underset{n\neq 0}{\mathrm{TE}_{nm}} \quad \frac{\Delta\gamma}{\gamma_{nm}} = \frac{n^2}{p_{nm}^2 - n^2}\,\frac{\epsilon - 1}{\epsilon(1 - \nu_{nm}^2)}\,\delta, \tag{8}$$

$$\mathrm{TE}_{om} \quad \frac{\Delta\gamma}{\gamma_{om}} = \frac{\epsilon - 1}{1 - \nu_{om}^2}\,\frac{p_{om}^2}{3}\,\delta^3. \tag{9}$$

In these equations p_{nm} is the mth root of $J_n(x) = 0$ for TM_{nm} waves and the mth root of $J_n'(x) = 0$ for TE_{nm} waves. Furthermore, $\nu_{nm} = \lambda/\lambda_{cnm}$ where $\lambda_{cnm} = 2\pi a/p_{nm}$.

We note that the change in propagation constant is of first order in δ for the TM_{nm} and TE_{nm} waves with $n \neq 0$, but of third order in δ for TE_{om} waves.

With $\epsilon = \epsilon' - j\epsilon''$ the perturbation of the propagation constant splits into phase perturbation $\Delta\beta$ and dielectric attenuation α_D. For a low loss dielectric we may assume $\epsilon'' \ll \epsilon'$ and get:

$$\begin{aligned}
&\mathrm{TM}_{nm} \quad \frac{\Delta\beta}{\beta_{nm}} = \frac{\epsilon' - 1}{\epsilon'}\,\delta,\\
&\underset{n\neq 0}{\mathrm{TE}_{nm}} \quad \frac{\Delta\beta}{\beta_{nm}} = \frac{n^2}{p_{nm}^2 - n^2}\,\frac{\epsilon' - 1}{\epsilon'(1 - \nu_{nm}^2)}\,\delta,\\
&\mathrm{TE}_{om} \quad \frac{\Delta\beta}{\beta_{om}} = \frac{p_{om}^2}{3}\,\frac{\epsilon' - 1}{1 - \nu_{om}^2}\,\delta^3,\\
&\mathrm{TM}_{nm} \quad \frac{\alpha_D}{\beta_{nm}} = \frac{\epsilon''}{\epsilon'^2}\,\delta,\\
&\underset{n\neq 0}{\mathrm{TE}_{nm}} \quad \frac{\alpha_D}{\beta_{nm}} = \frac{n^2}{p_{nm}^2 - n^2}\,\frac{\epsilon''}{\epsilon'^2(1 - \nu_{nm}^2)}\,\delta,\\
&\mathrm{TE}_{om} \quad \frac{\alpha_D}{\beta_{nm}} = \frac{p_{om}^2}{3}\,\frac{\epsilon''}{1 - \nu_{om}^2}\,\delta^3.
\end{aligned} \tag{10}$$

Unfortunately the range in which (7), $\cdots$ (10) are good approximations is rather limited. Actually

$$\frac{1 - \nu_{nm}^2}{\nu_{nm}} \cdot \frac{2\pi a}{\lambda} \cdot \frac{\Delta\beta}{\beta_{nm}}$$

has to be small compared to unity, at least not larger than 0.1. In the

case of the TM_{11} wave in a 2-inch pipe at $\lambda = 5.4$ mm this limit is reached with $\Delta\beta/\beta_{11} = 0.45 \times 10^{-3}$ or $\delta = 0.75 \times 10^{-3}$ with $\epsilon = 2.5$.

To evaluate (1) beyond this limit we may use the approximations:

$$\begin{aligned} \frac{1}{\rho x} \frac{W_n(x, \rho x)}{U_n(x, \rho x)} &= \cot (1 - \rho)x = \cot \delta x, \\ \frac{1}{\rho x} \frac{V_n(x, \rho x)}{Z_n(x, \rho x)} &= - \tan (1 - \rho) = - \tan \delta x. \end{aligned} \tag{11}$$

The approximations (11) require $x \gg n$, and $\delta x < \pi/2$, as shown in Appendix I. These requirements are usually satisfied for the lower order modes in a multimode waveguide with a thin dielectric layer.

Equation (1) has been evaluated for the TM_{11} mode using the approximations (11) and a method which is described in Appendix I. The results are shown in Fig. 2 for $\epsilon = 2.5$ and several values of a/λ.

In introducing a dielectric coat we have to be aware of the change in attenuation constant the TE_{01} mode will suffer. Not only the loss factor of the dielectric material will cause additional losses, but the concentration of more field energy into the dielectric will increase the wall currents and so the wall current losses.

To calculate the wall current attenuation of the TE_{01} mode in the round waveguide with the dielectric layer, we proceed in the usual

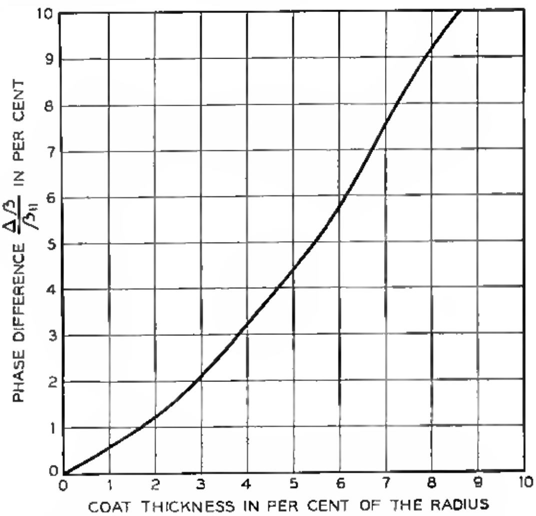

Fig. 2(a) — Change in phase constant of the TM_{11} wave in the dielectric-coated waveguide. $\epsilon = 2.5$; $a/\lambda = 1.03$.

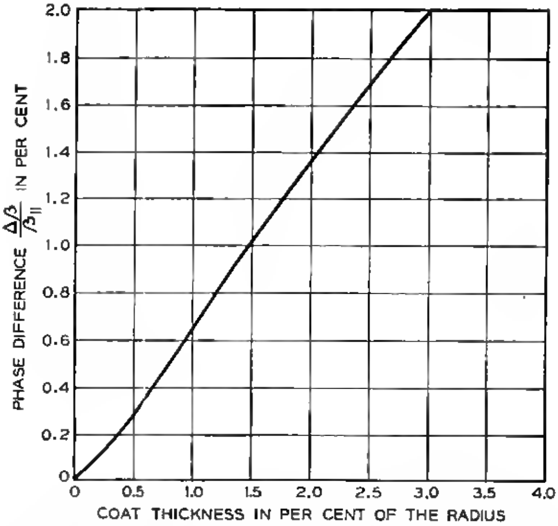

Fig. 2(b) — Change in phase constant of the TM_{11} wave in the dielectric-coated waveguide. $\epsilon = 2.5$; $a/\lambda = 2.06$.

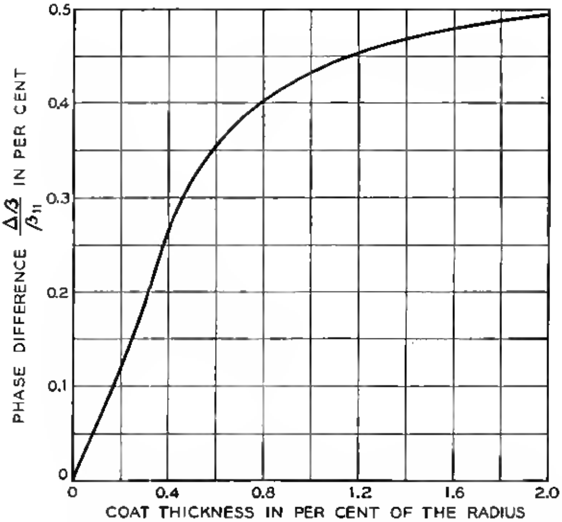

Fig. 2(c) — Change in phase constant of the TM_{11} wave in the dielectric-coated waveguide. $\epsilon = 2.5$; $a/\lambda = 4.70$.

manner. Assuming the conductivity to be high though finite and the loss factor of the dielectric material to be small, we take the field pattern and wall currents of the lossless case and calculate the total transmitted power P and the power P_M absorbed per unit length of the waveguide by the metal walls of finite conductivity. The wall current attenuation α_M is then given by

$$\alpha_M = \frac{1}{2}\frac{P_M}{P}. \tag{12}$$

The result of this calculation as carried through in Appendix II is:

$$\frac{\alpha_M}{\alpha_{om}} = \frac{x_2^2}{p_{om}^2}\sqrt{\frac{x_2^2 - x_1^2}{x_2^2 - \epsilon x_1^2}}\left\{1 + \frac{\pi^2}{2}\rho x_2\left[\frac{x_2^2}{x_1^2} - 1\right]\right. \cdot \left.\left[U_1(x_2, \rho x_2) - \frac{\rho x_2}{2}R_1(x_2, \rho x_2)\right]R_1(x_2, \rho x_2)\right\}^{-1}. \tag{13}$$

In this expression α_M is related to the TE_{om} attenuation constant α_{om} of the plain waveguide.

For $1 - \rho = \delta \ll 1$ we introduce the series expansions of the functions $U_1(x_2, \rho x_2)$ and $R_1(x_2, \rho x_2)$,

$$\frac{\Delta\alpha_M}{\alpha_{om}} = (\epsilon - 1)\frac{p_{om}^2}{\nu_{om}^2}\delta^2. \tag{14}$$

Here $\Delta\alpha_M$ is defined as the change in wall current attenuation compared to the attenuation in the plain waveguide.

III. PROPERTIES OF COUPLED TRANSMISSION LINES

Wave propagation in gentle bends of a round waveguide can be described in terms of normal modes of the straight guide.[1] The bend causes coupling between the normal modes. The TE_{01} wave couples to the TM_{11} wave and to the TE_{1n} waves and the propagation in the bend is described by an infinite set of simultaneous linear differential equations. An adequate approximate treatment is to consider only coupling between TE_{01} and one of the spurious modes at a time. Furthermore, only the forward waves need to be considered, since the relative power coupled from the forward waves into the backward waves is quite small. Thus, the infinite set of equations reduces to the well known coupled line equations:[2]

$$\begin{aligned} \frac{dE_1}{dz} + \gamma_1 E_1 - jcE_2 &= 0, \\ \frac{dE_2}{dz} + \gamma_2 E_2 - jcE_1 &= 0, \end{aligned} \tag{15}$$

in which

$E_{1,2}(z)$ = wave amplitudes in mode 1 (here always TE_{01}) and mode 2 (TM_{11} or one of the TE_{1n}), respectively;

$\gamma_{1,2}$ = propagation constant of mode 1 and 2, respectively (the small perturbation of $\gamma_{1,2}$ caused by the coupling may be neglected here); and

c = coupling coefficient between modes 1 and 2.

Subject to the initial conditions:

$$E_1(0) = 1, \qquad E_2(0) = 0,$$

the solution of (15) is:

$$\begin{aligned} E_1 &= \frac{1}{2}\left[1 + \frac{\Delta\gamma}{\sqrt{\Delta\gamma^2 - 4c^2}}\right] e^{-\Gamma_1 z} + \frac{1}{2}\left[1 - \frac{\Delta\gamma}{\sqrt{\Delta\gamma^2 - 4c^2}}\right] e^{-\Gamma_2 z}, \\ E_2 &= \frac{jc}{\sqrt{\Delta\gamma^2 - 4c^2}} [e^{-\Gamma_2 z} - e^{-\Gamma_1 z}], \end{aligned} \tag{16}$$

where $\Delta\gamma = \gamma_1 - \gamma_2$ and $\Gamma_{1,2} = \frac{1}{2}[\gamma_1 + \gamma_2 \pm \sqrt{\Delta\gamma^2 - 4c^2}]$. Γ_1 and Γ_2 are the propagation constants of the two coupled line normal modes. Both coupled line normal modes are excited by the initial conditions. For $|c/\Delta\gamma| \ll 1$, equations (16) can be simplified:

$$\begin{aligned} E_1 &= \left[1 - 2\frac{c^2}{\Delta\gamma^2} \sinh\frac{1}{2}\Delta\gamma z\, e^{\frac{1}{2}\Delta\gamma z}\right] e^{-(\gamma_1 - (c^2/\Delta\gamma))z}, \\ E_2 &= j\frac{2c}{\Delta\gamma} \sinh\frac{1}{2}\Delta\gamma z\, e^{-\frac{1}{2}(\gamma_1+\gamma_2)z}. \end{aligned} \tag{17}$$

We are concerned with a difference in phase constant which is much larger than the difference in attenuation constant. Consequently in $\Delta\gamma = j\Delta\beta + \Delta\alpha$ we have $|\Delta\beta| \gg |\Delta\alpha|$ and we may write:

$$\begin{aligned} E_1 &= \left[1 + j\frac{2c^2}{\Delta\beta^2} \sin\frac{1}{2}\Delta\beta z e^{j\frac{1}{2}\Delta\beta z}\right] \\ &\qquad \cdot\exp\left[-j\left(\beta_1 + \frac{c^2}{\Delta\beta^2}\Delta\beta\right)z - \left(\alpha_1 - \frac{c^2}{\Delta\beta^2}\Delta\alpha\right)z\right], \\ E_2 &= j\frac{2c}{\Delta\beta}\cdot\sin\frac{1}{2}\Delta\beta z \exp\left[-j\frac{1}{2}(\beta_1 + \beta_2)z - \frac{1}{2}(\alpha_1 + \alpha_2)z\right]. \end{aligned} \tag{18}$$

We note that the amplitude E_1, apart from suffering an attenuation, varies in an oscillatory manner, the maximum being $E_1 = 1$ and the minimum $E_1 = 1 - 2(c^2/\Delta\beta^2)$. Accordingly, the maximum mode conversion loss is given by:

$$20 \log_{10} \frac{E_{1\max}}{E_{1\min}} = 17.35 \frac{c^2}{\Delta\beta^2}. \tag{19}$$

The attenuation constant of E_1 is modified by the presence of the coupled wave. Compared to the uncoupled attenuation constant it has been changed by

$$\frac{\Delta\alpha_c}{\alpha_1} = \frac{c^2}{\Delta\beta^2}\left(\frac{\alpha_2}{\alpha_1} - 1\right). \tag{20}$$

The amplitude E_2 varies sinusoidally. From our point of view it is an unwanted mode. The power level compared to the E_1 power is

$$20 \log_{10} \frac{E_{2\max}}{E_1} = 20 \log_{10} \frac{2c}{\Delta\beta}. \tag{21}$$

So far we have considered only a constant value of the coupling coefficient, c, corresponding to a uniform bend. The attenuation in such a uniform bend is increased according to (20), and the worst condition we can encounter at the end of the bend is a mode conversion loss (19) and a spurious mode level (21).

A practical case of changing curvature and consequently changing coupling coefficient is the serpentine bend. A waveguide with equally spaced supports deforms into serpentine bends under its own weight. The curve between two particular supports is well known from the theory of elasticity. An analysis of circular electric wave transmission through serpentine bends[6] shows that mode conversion becomes seriously high at certain critical frequencies when the supporting distance is a multiple of the beat wavelength between the TE_{01} and a particular coupled mode. The beat wavelength is here defined as

$$\lambda_b = \frac{2\pi}{\Delta\beta}. \tag{22}$$

In serpentine bends formed by elastic curves, mode conversion at the critical frequencies causes an increase in TE_{01} attenuation[6]

$$\frac{\Delta\alpha_s}{\alpha_{01}} = -\left[\frac{w}{EI}\frac{c_0}{\Delta\beta^2\alpha_{01}}\right]^2 \frac{\alpha_{01}}{\Delta\alpha}, \tag{23}$$

and a spurious mode level in the particular coupled mode[6]

$$\left|\frac{E_2}{E_1}\right| = \frac{w}{EI}\frac{c_0}{\Delta\beta^2\alpha_{01}}\left|\frac{\alpha_{01}}{\Delta\alpha}\right|, \tag{24}$$

where w = weight per unit length of the pipe,
E = modulus of elasticity,
I = moment of inertia,

and c_0 is the factor in the bend coupling coefficient $c = c_0/R$ determined by waveguide dimensions, frequency and the particular mode. R is the bend radius.

Equations (23) and (24) hold only as long as

$$4\Delta\alpha_s \ll |\Delta\alpha| \tag{25}$$

is satisfied. The rate of conversion loss has to be small compared to the difference between the rates of decay for the unwanted mode and TE_{01} amplitudes. If (25) is not satisfied a cyclical power transfer between TE_{01} and the particular coupled mode occurs, and the TE_{01} transmission is seriously distorted.

Another case of changing curvature of the waveguide is random deviation from straightness, which must be tolerated in any practical line. Such deviations from straightness change the curvature only very gradually, and since there is no coupled wave in the dielectric coated guide, which has the same phase constant as the TE_{01} wave, the curvature may be assumed to vary only slowly compared to the difference in phase constants. Under this condition[7] the normal mode of the straight waveguide, which here is the TE_{01} mode, will be transformed along the gradually changing curvature into the normal mode of the curved waveguide, which is a certain combination of TE_{01} and the coupled modes. This normal mode will always be maintained and no spurious modes will be excited.

The change in transmission loss is therefore given alone by the difference between the attenuation constants of the TE_{01} wave and the normal mode of the curved waveguide. The propagation constants of the normal modes of the coupled lines are Γ_1 and Γ_2 as given by (16). Only the mode with Γ_1 will be excited here and consequently the attenuation difference is given by (20).

IV. CIRCULAR ELECTRIC WAVE TRANSMISSION THROUGH CURVED SECTIONS OF THE DIELECTRIC-COATED GUIDE

In curved sections of the plain waveguide the wave solution has been described in terms of the normal modes of the straight waveguide. In this presentation it has been found that the TE_{01} wave couples to the TM_{11} and TE_{1n} waves in gentle bends. Likewise, the wave solution in curved sections of the dielectric-coated guide can be described in terms of the normal waves of the straight dielectric-coated guide. In a waveguide with a thin dielectric coat the normal waves may be considered as perturbed normal modes of the plain waveguide. Consequently the bend solution of the plain waveguide may be taken as the first order approximation

for the bend solution of the dielectric-coated guide. In this first order approximation the TE_{01} wave of the dielectric-coated guide couples to the TM_{11} and TE_{1n} waves of the dielectric-coated guide and the coupling coefficients are the same as in the bend-solution of the plain waveguide:[1]

$$TE_{01} \rightleftarrows TM_{11} \qquad c = 0.18454 \frac{\beta a}{R},$$

$$TE_{01} \rightleftarrows TE_{11}$$

$$c = \left[\frac{0.09319(\beta a)^2 - 0.84204}{\sqrt{\beta_{01} a \beta_{11} a}} + 0.09319\sqrt{\beta_{01} a \beta_{11} a}\right]\frac{1}{R},$$

$$TE_{01} \rightleftarrows TE_{12} \tag{26}$$

$$c = \left[\frac{0.15575(\beta a)^2 - 3.35688}{\sqrt{\beta_{01} a \beta_{12} a}} + 0.15575\sqrt{\beta_{01} a \beta_{11} a}\right]\frac{1}{R},$$

$$TE_{01} \rightleftarrows TE_{13}$$

$$c = \left[\frac{0.01376(\beta a)^2 - 0.60216}{\sqrt{\beta_{01} a \beta_{13} a}} + 0.01376\sqrt{\beta_{01} a \beta_{13} a}\right]\frac{1}{R},$$

β = free-space phase constant.

With the coupling coefficients (26) and the propagation constants as given by (10) and (14) the coupled line equations can be used to compute the TE_{01} transmission through curved sections. Since the absolute value of $c/\Delta\gamma$ is usually small compared to unity, the mode conversion loss is given by (19), the increase in TE_{01} attenuation by (20), and the spurious mode level by (21).

The curves in Figs. 3, 4 and 5 have been calculated for the 2-inch pipe at a wavelength of 5.4 mm. They take into account the effects of the three most seriously coupled modes, TM_{11}, TE_{11}, and TE_{12}.

In very gentle bends of a guide with a very thin coat, coupling effects to the TE_{1m} modes are small and only TM_{11} coupling influences the TE_{01} transmission. The increase in TE_{01} attenuation in such bends is obtained by substituting the expression (10) for the TM_{11} phase difference in (20).

$$\frac{\Delta\alpha_c}{\alpha_{01}} = \frac{0.034}{\nu_{01}^2} \frac{\epsilon'^2}{(\epsilon' - 1)^2} \frac{1}{\delta^2} \frac{a^2}{R^2}. \tag{27}$$

Note that (27) requires $|c/\Delta\gamma| \ll 1$ and consequently $(1/\delta)(a/R) \ll 1$. In Fig. 4, as well as in (27), losses in the dielectric coat have been neglected. Equations (10) show that the dielectric losses are small compared to the wall current losses for a low loss dielectric coat.

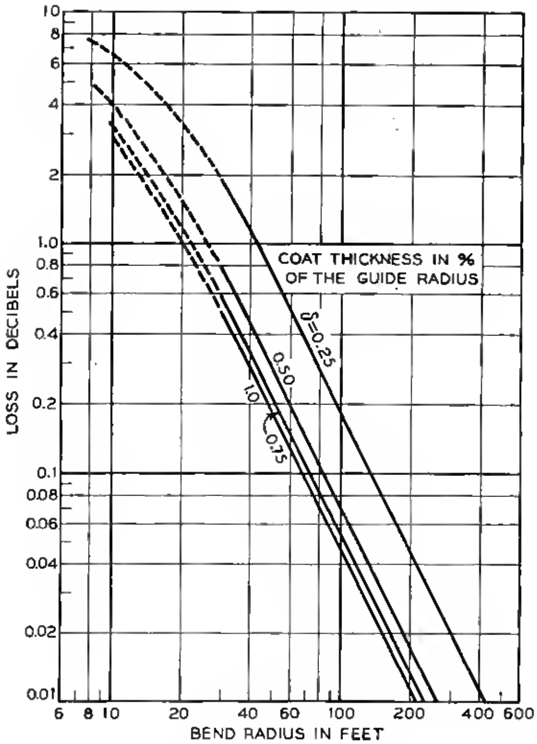

Fig. 3 — Maximum conversion loss of the TE_{01} wave in a uniform bend of the dielectric-coated waveguide. $2a = 2$ inches; $\lambda = 5.4$ mm; $\epsilon = 2.5$.

There are two different applications of the dielectric-coated guide, which require different designs:

1. Intentional bends

These are relatively short sections and the increase in TE_{01} attenuation is usually small compared to the bend loss. Also, a high spurious mode level can be tolerated, because with a mode filter at the end of the bend we can always control the spurious mode level. Consequently, the only limit set for this type of bend is the mode conversion loss of the TE_{01} wave. There is conversion loss mainly to TE_{11}, TE_{12} and TM_{11}. Increasing the phase difference of the TM_{11} wave by making the dielectric coat thicker decreases the phase-difference between TE_{01} and TE_{12} and increases the phase-difference between TE_{01} and TE_{11}. Apparently we get

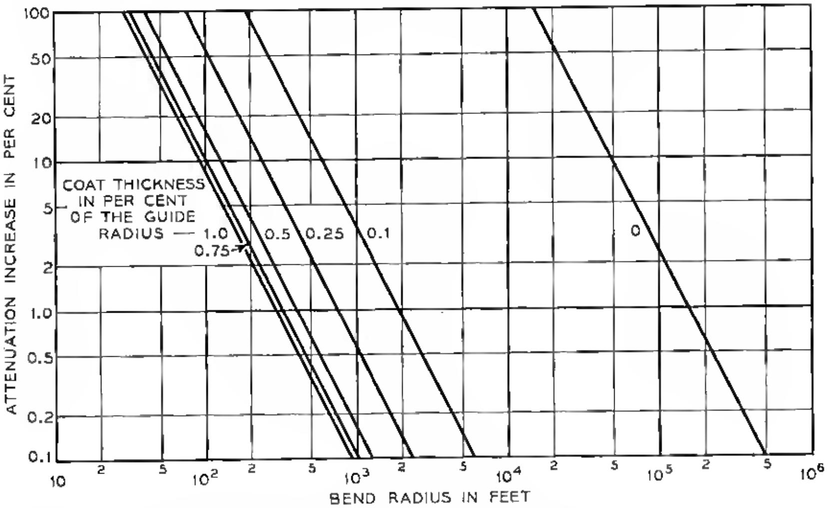

Fig. 4 — Increase in TE_{01} attenuation of a dielectric-coated guide in a uniform bend. $2a = 2$ inches; $\epsilon = 2.5$; $\lambda = 5.4$ mm.

very near to an optimum design with a dielectric coat for which:

$$\left(\frac{\Delta\beta}{c}\right)_{TM_{11}} = \left(\frac{\Delta\beta}{c}\right)_{TE_{12}}. \tag{28}$$

For this condition conversion losses to TM_{11} and TE_{12} are equal, while the conversion loss to TE_{11} is small.

To find values which satisfy (28) we will generally have to solve (1) because the TM_{11} phase difference required by (28) is too large to be calculated with the first order approximation. At a wavelength of $\lambda = 5.4$ mm and a dielectric constant $\epsilon = 2.5$ the optimum thickness of the coat according to (28) is $\delta = 1.25$ per cent in the 2-inch pipe. In Fig. 6 the mode conversion loss in a dielectric-coated guide of this design is plotted versus bending radius.

2. Random deviations from straightness

As mentioned before, random deviations from straightness change the curvature only gradually and only one normal mode propagates. Mode conversion loss and spurious mode level are very low. The normal mode attenuation depends on the curvature. The increase in normal mode attenuation as caused by curvature is obtained by adding the attenuation terms (20) of the various straight guide modes which are contained

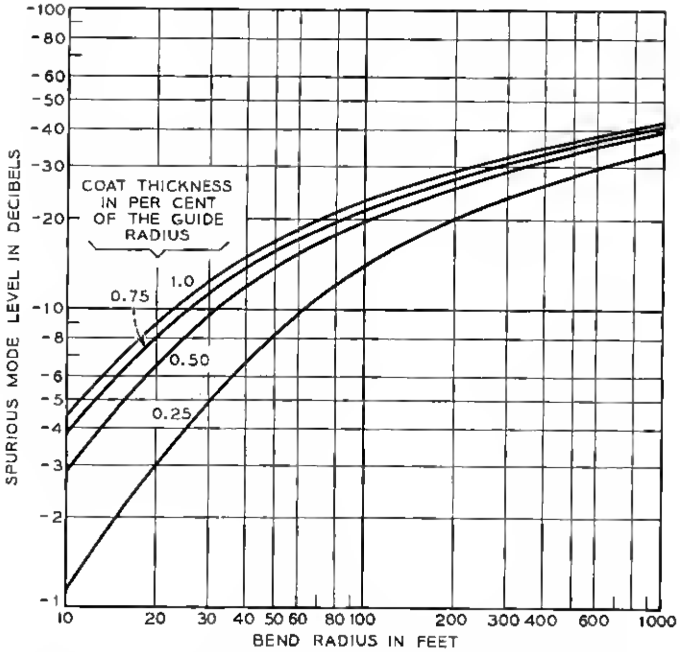

Fig. 5 — Spurious mode level in a uniform bend of a dielectric-coated guide. $2a = 2$ inches; $\epsilon = 2.5$; $\lambda = 5.4$ mm.

in the normal mode. The bending radius R is a function of position and an average bending radius,

$$\frac{1}{R_{\mathrm{Av}}^2} = \frac{1}{z} \int_0^z \frac{dz}{R^2}, \tag{29}$$

has to be used in (20).

All the attenuation terms (20) decrease with increasing coat thickness δ; the TE_{01} attenuation in the straight guide (10) and (14) increases with δ. Consequently there is an optimum thickness for which the total increase in attenuation is a minimum. This optimum thickness depends on the average radius of curvature. In Fig. 7 optimum thickness and the corresponding increase in attenuation have been plotted versus the average radius of curvature.

In gentle curvatures the normal mode is a mixture of TE_{01} and TM_{11} only and the attenuation increase as caused by the curvature is given by (27). In this case the optimum thickness and the corresponding attenuation increase are:

$$\delta_{\text{opt}} = \frac{1}{\sqrt[4]{2}} \frac{1}{p_{01}} \frac{\sqrt{\epsilon'}}{(\epsilon' - 1)^{3/4}} \sqrt{\frac{a}{R_{\text{Av}}}}, \tag{30}$$

$$\frac{\Delta\alpha}{\alpha_{01}} = \frac{\sqrt{2}}{\nu_{01}^{2}} \frac{\epsilon'}{\sqrt{\epsilon' - 1}} \frac{a}{R_{\text{Av}}}. \tag{31}$$

We note that δ_{opt} does not depend on frequency.

So far we have listed only the useful properties of the dielectric-coated guide. There is, however, one serious disadvantage. Serpentine bends caused by equally spaced discrete supports and the elasticity of the pipe are inherently present in any waveguide line. At critical frequencies, when the supporting distance l is a multiple of the beat wavelength λ_b between TE_{01} and a particular coupled mode, an increase in TE_{01} attenuation (23) and a spurious mode level (24) result from the mode conversion.

We evaluate (23) and (24) for a dielectric-coated copper pipe of 2.000-inch I.D. and 2.375-inch O.D. and a supporting distance l = 15 ft. The

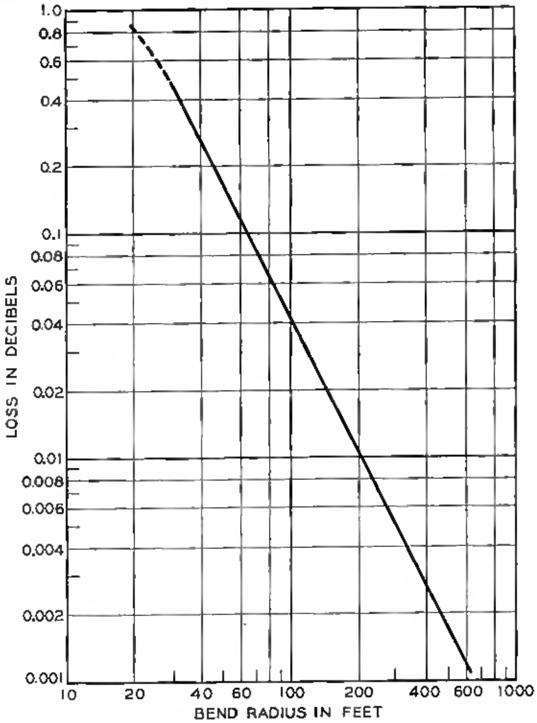

Fig. 6 — Bend conversion loss in a dielectric-coated guide of optimum design according to equation (28). $2a$ = 2 inches; δ = 1.25%; ϵ = 2.5; λ = 5.4 mm.

result is for coat thicknesses which cause the wavelength $\lambda = 5.4$ mm to be critical with respect to TE_{01}–TM_{11} coupling:

$$\begin{aligned} \delta &= 0.002 \quad & l &= \lambda_b \quad & \frac{\Delta\alpha_s}{\alpha_{01}} &= 43.3 \quad & 20 \log_{10} \left| \frac{E_2}{E_1} \right| &= -1.29\ db \\ &= 0.004 & &= 2\lambda_b & &= 2.70 & &= -13.32\ db \\ &= 0.006 & &= 3\lambda_b & &= 0.169 & &= -25.37\ db. \end{aligned}$$

The values corresponding to $\delta = 0.002$ do not satisfy the condition (25). Therefore they cannot be considered as a quantitative result but only as an indication that the mode conversion is very high. We conclude from these values that the mode conversion is much too high for a coat thickness which makes the beat wavelength between TE_{01} and TM_{11} equal to the supporting distance or half of it.

If no other measures can be taken, such as removing the periodicity of the supports or inserting mode filters, the lowest critical frequencies of TE_{01}–TM_{11} coupling bave to be avoided. A dielectric coat must be

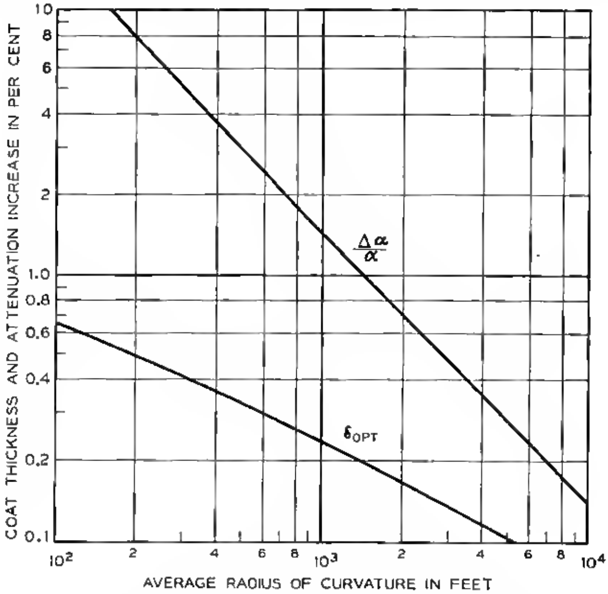

Fig. 7 — Optimum coat thickness and increase in TE_{01} attenuation of a dielectric-coated guide with random deviations from straightness. $2a = 2$ inches; $\lambda = 5.4$ mm; $\epsilon = 2.5$.

chosen, which is thin enough or thick enough to keep these critical frequencies out of the band.

Mode conversion from TE_{01} to the TE_{1m} modes in serpentine bends is not changed substantially by the presence of the dielectric coat. Generally these mode conversion effects decrease rapidly with the beat wavelength. When the curvature varies slowly compared to the difference in phase constant between coupled waves almost pure normal mode propagation is maintained.[7]

V. MODE CONVERSION AT TRANSITIONS FROM PLAIN WAVEGUIDE TO THE DIELECTRIC-COATED WAVEGUIDE

We consider a round waveguide, a section of which has a dielectric layer next to the walls. A pure TE_{01} wave incident on this dielectric-coated section will excite TE_{om} waves. For circular electric wave transmission it is important to keep low the power level of the higher circular electric waves which have low loss.

In an evaluation of Schelkunoff's generalized telegraphist's equations for the TE_{01} mode in a circular waveguide containing an inhomogeneous dielectric[1] S. P. Morgan describes the wave propagation in the dielectric-filled waveguide in terms of normal modes of the unfilled waveguide. The only restriction made in this analysis for the dielectric insert is

$$\frac{1}{S}\int_S |\epsilon - 1|\, dS \ll 1, \tag{32}$$

where S is the cross-sectional area of the guide.

The dielectric-coated guide satisfies (32), and we may use the results of Morgan's evaluation here.

The round waveguide is considered as an infinite set of transmission lines, each of which represents a normal mode. Along the dielectric-coated section the TE_{0m} transmission lines are coupled mutually. The coupling coefficient d between TE_{01} and one of the TE_{0m} waves is obtained by taking Morgan's general formula and evaluating it for the dielectric coat:

$$d = p_{01}p_{0m}\frac{(\epsilon' - 1)\beta^2}{\sqrt{\beta_{01}\beta_{0m}}}\frac{\delta^3}{3}. \tag{33}$$

We introduce this coupling coefficient into the coupled line equations (16). Since $d/\Delta\beta \ll 1$ for any of the coupled modes, the spurious mode level at the output of the dielectric-coated section is given by (21),

$$20\log_{10}\left|\frac{E_{2\max}}{E_1}\right| = 20\log_{10}\frac{2}{3}\frac{(\epsilon' - 1)p_{01}p_{0m}\beta^2}{(\beta_{01} - \beta_{om})\sqrt{\beta_{01}\beta_{0m}}}\delta^3. \tag{34}$$

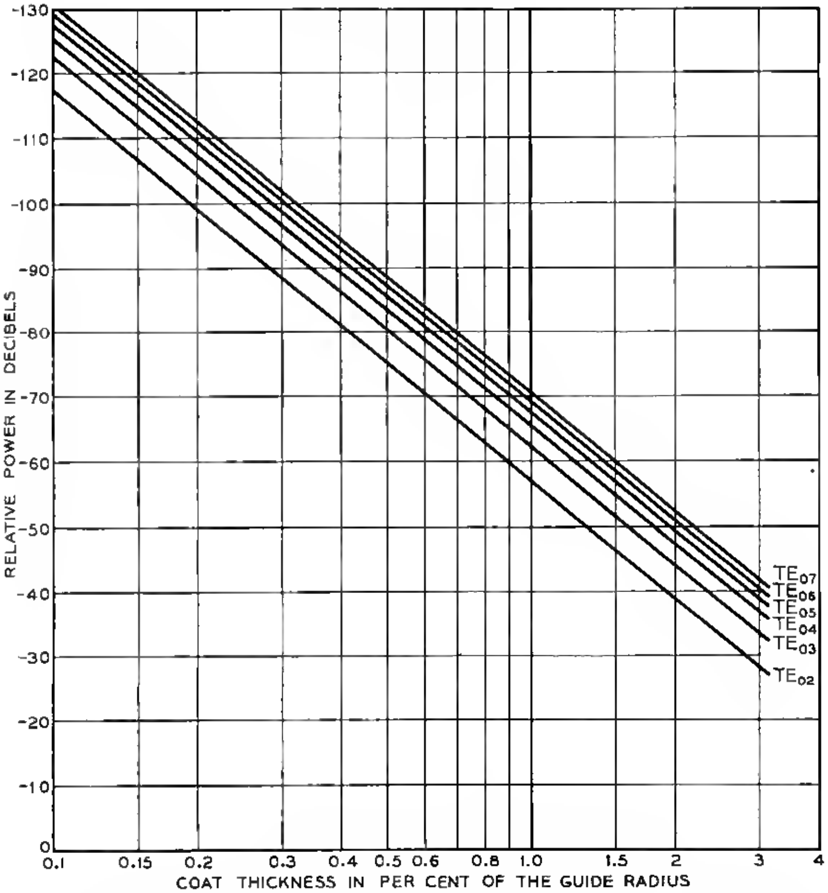

Fig. 8 — Higher circular electric waves at a transition from plain waveguide to the dielectric-coated guide. $\epsilon = 2.5$; $a/\lambda = 4.70$.

Fig. 8 shows an evaluation of (34) for $a/\lambda = 4.70$ and $\epsilon' = 2.5$ corresponding to a 2-inch pipe with a polystyrene coat at $\lambda = 5.4$ mm.

VI. SUMMARY

A theoretical analysis of wave propagation in the dielectric-coated guide is presented to provide information necessary for circular electric wave transmission in this waveguide structure. The normal modes of a waveguide with a thin dielectric coat are perturbed modes of the plain waveguide. While the perturbation of the phase constant is only of third order of the coat thickness for the circular electric waves, it is of first

order for all other modes. Thus, the degeneracy of equal phase constants of circular electric waves and TM_{1m} waves can be removed quite effectively. The additional attenuation to TE_{01} wave as caused by the dielectric loss in the coat and the increased wall current loss remains small as long as the coat is thin.

The dielectric-coated guide may be used for negotiating intentional bends or for avoiding extreme straightness requirements on normally straight sections. For intentional bends of as small a radius as possible, an optimum thickness of the coat makes the mode-conversion losses to TM_{11} and TE_{12} modes equal and minimizes the total conversion loss. For random deviations from straightness, an average radius of curvature is defined. For this average radius an optimum thickness of the coat minimizes the additional TE_{01} attenuation as caused by curvature and dielectric coat. In random deviations from straightness, propagation of only one normal mode is maintained as long as only the rate of change of curvature is small compared to the square of the difference in phase constant between TE_{01} and any coupled mode.

Serpentine bends, caused by equally spaced supports and the associated elastic deformation of the pipe, increase the TE_{01} attenuation substantially at certain critical frequencies, when the supporting distance is a multiple of the beat wavelength. The lowest critical frequencies of TE_{01}–TM_{11} coupling corresponding to a beat wavelength, which is equal to the supporting distance or half of it, have to be avoided by choosing the proper coat thickness.

At transitions from plain waveguide to dielectric-coated guide higher circular electric waves are excited by the TE_{01} wave. However, the power level of these spurious modes is low for a thin dielectric coat.

ACKNOWLEDGMENTS

The dielectric-coated waveguide is the subject of two patents.[9] Some of its useful properties were brought to the writer's attention by a communication between the Standard Telecommunication Laboratory, Ltd., England, and S. E. Miller. For helpful discussions the writer is indebted to E. A. J. Marcatili, S. E. Miller, and D. H. Ring.

APPENDIX I

Approximate Solutions of the Characteristic Equation

In the following calculation we will use the definitions:

$$\begin{aligned} R_n(x, \rho x) &= J_n(x)\, N_{n-1}(\rho x) - J_{n-1}(\rho x) N_n(x), \\ S_n(x, \rho x) &= J_{n-1}(x)\, N_n(\rho x) - J_n(\rho x)\, N_{n-1}(x), \end{aligned} \tag{35}$$

and their relation to the definitions (2);

$$\begin{aligned} \frac{W_n(x, \rho x)}{U_n(x, \rho x)} &\equiv \rho x \frac{R_n(x, \rho x)}{U_n(x, \rho x)} + n \equiv -\rho x \frac{S_{n+1}(x, \rho x)}{U_n(x, \rho x)} - n, \\ \frac{V_n(x, \rho x)}{Z_n(x, \rho x)} &\equiv \rho x \frac{x U_{n-1}(x, \rho x) + n R_n(x, \rho x)}{x S_n(x, \rho x) + n U_n(x, \rho x)} + n. \end{aligned} \tag{36}$$

To find solutions of (1) for $1 - \rho \equiv \delta \ll 1$ we first substitute for the functions (36) their series expansions with respect to δ.

We have for instance

$$U_n(x, \rho x) = N_n(x)\, J_n(x - \delta x) - J_n(x)\, N_n(x - \delta x)$$

where we introduce

$$\begin{aligned} J_n(x - \delta x) &= J_n(x) + \delta x J_n'(x) + \frac{(\delta x)^2}{2} J_n''(x) + \cdots \\ &= J_n(x) + \delta x \left[J_{n+1}(x) - \frac{n}{x} J_n(x) \right] \\ &\quad + \frac{(\delta x)^2}{2} \left[\frac{1}{x} J_{n+1}(x) + \left(\frac{n^2 - n}{x^2} - 1 \right) J_n(x) \right] + \cdots, \end{aligned}$$

and

$$\begin{aligned} N_n(x - \delta x) &= N_n(x) + \delta x \left[N_{n+1}(x) - \frac{n}{x} N_n(x) \right] \\ &\quad + \frac{(\delta x)^2}{2} \left[\frac{1}{x} N_{n+1}(x) + \left(\frac{n^2 - n}{x^2} - 1 \right) N_n(x) \right] + \cdots. \end{aligned}$$

Upon using the relation

$$J_{n+1}(x) N_n(x) - J_n(x) N_{n+1}(x) = \frac{2}{\pi x}$$

we get

$$U_n(x, \rho x) = \frac{2}{\pi} \delta \left[1 + \frac{\delta}{2} + \frac{\delta^2 x^2}{6} \left(\frac{n^2 + 2}{x^2} - 1 \right) \right] + 0(\delta^4), \tag{37}$$

and by the same procedure

$$\begin{aligned} R_n(x, \rho x) = \frac{2}{\pi x} \Bigg[1 - (n - 1)\delta &+ \left(\frac{(n-1)(n-2)}{x^2} - 1 \right) \frac{(\delta x)^2}{2} \\ &- \frac{n-2}{x} \left(1 + \frac{(n-1)(n-3)}{x^3} \right) \frac{(\delta x)^3}{6} \Bigg] + 0(\delta^4). \end{aligned} \tag{38}$$

With these expressions the functions (36) are approximated by:

$$\begin{aligned} \frac{W_n(x, \rho x)}{U_n(x, \rho x)} &= \frac{1}{\delta}\left(1 - \frac{\delta}{2}\right) + 0(\delta), \\ \frac{V_n(x, \rho x)}{Z_n(x, \rho x)} &= \delta(x^2 - n^2) + 0(\delta^2), \end{aligned} \tag{39}$$

and for $n = 0$ especially:

$$\begin{aligned} \frac{1}{\rho x^2}\frac{V_0(x, \rho x)}{Z_0(x, \rho x)} &= -\frac{1}{x}\frac{U_1(x, \rho x)}{R_1(x, \rho x)} \\ &= -\delta\left[1 + \frac{\delta}{2} + \delta^2\left(\frac{1}{3}x^2 + \frac{1}{2}\right)\right] + 0\,(\delta^4)\,. \end{aligned} \tag{40}$$

Since for $\delta = 0$ the roots of $J_n(\rho x_1) = 0$ and $J_n'(\rho x_1) = 0$ represent the solutions of (1) for the TM and TE waves respectively, we expand the Bessel functions of the argument ρx_1 in series around these roots:

$$\rho x_1 = (1 - \delta)(p_{nm} + \Delta x).$$

The result for TM waves is

$$J_n(p_{nm}) \equiv 0: \qquad \frac{J_n'(\rho x_1)}{J_n(\rho x_1)} = \frac{1}{\Delta x - \delta p_{nm}}; \tag{41}$$

for TE waves:

$$J_n'(p_{nm}) \equiv 0: \qquad \frac{J_n'(\rho x_1)}{J_n(\rho x_1)} = \frac{n^2 - p_{nm}^2}{p_{nm}^2}(x - \delta p_{nm});$$

and for TE_{om} waves especially:

$$\begin{aligned} &J_0'(p_{om}) = 0: \\ &-\frac{J_0'(\rho x_1)}{J_0(\rho x_1)} = \Delta x - \delta p_{om} - \frac{1}{2p_{om}}(\Delta x^2 + \delta^2 p_{om}^2) - \frac{\delta^3}{6}p_{om}(3 + 2p_{om}^2). \end{aligned} \tag{42}$$

Introducing (39 $\cdots$ 42) into the characteristic equation (1) and neglecting higher order terms in δ and Δx we get the following approximations for (1):

$$\begin{aligned} TM_{nm} \text{ waves:} &\qquad \Delta x = \left(p_{nm} - \frac{1}{\epsilon}\frac{x_2^2}{p_{nm}}\right)\delta \\ \underset{n \neq 0}{TE_{nm}} \text{ waves:} &\qquad \Delta x = \frac{x_2^2 - p_{nm}^2}{p_{nm}^2 - n^2}\frac{n^2}{\epsilon p_{nm}}\delta \end{aligned} \tag{43}$$

$$TE_{om} \text{ waves:} \qquad \Delta x = -\frac{p_{om}}{3}(x_2^2 - p_{om}^2)\delta^3. \tag{44}$$

If we write for the perturbed propagation constant

$$\gamma = \gamma_{nm} + \Delta\gamma,$$

we get with (3) and (4):

$$\Delta x = a^2 \frac{\gamma_{nm}}{p_{nm}} \Delta\gamma$$

$$= ja \frac{\sqrt{1 - \nu_{nm}^2}}{\nu_{nm}} \Delta\gamma,$$

and

$$x_2^2 = (\epsilon - 1 + \nu_{nm}^2) \frac{p_{nm}^2}{\nu_{nm}^2},$$

in which

$$\nu_{nm} = \frac{\lambda}{\lambda_{cnm}} = \frac{p_{nm}}{2\pi} \frac{\lambda}{a}.$$

Using these expressions in (43) and (44) we finally get approximate formulae for the perturbation of the propagation constant as caused by the dielectric coat:

$$\text{TM}_{nm} \text{ waves:} \qquad \frac{\Delta\gamma}{\gamma_{nm}} = \frac{\epsilon - 1}{\epsilon}\delta$$

$$\underset{n \neq 0}{\text{TE}_{nm} \text{ waves:}} \qquad \frac{\Delta\gamma}{\gamma_{nm}} = \frac{n^2}{p_{nm}^2 - n^2} \frac{\epsilon - 1}{\epsilon} \frac{1}{1 - \nu_{nm}^2}\delta \tag{45}$$

$$\text{TE}_{0m} \text{ waves:} \qquad \frac{\Delta\gamma}{\gamma_{om}} = \frac{p_{om}^2}{3} \frac{\epsilon - 1}{1 - \nu_{om}^2}\delta^3.$$

The series expansions used so far hold only when $(1 - \rho)x \ll 1$. Approximate expressions which require only $(1 - \rho) \ll 1$ are:[8]

$$U_n(x, \delta x) = \frac{2}{\pi} \frac{\sin \dfrac{1 - \rho}{\sqrt{\rho}} \sqrt{\rho x^2 - n^2}}{\sqrt{\rho x^2 - n^2}},$$

$$S_n(x, \delta x) = \frac{2}{\pi} \frac{1}{(1 - \rho)x} \left[\sqrt{\rho} \cos\left(\frac{1 - \rho}{\sqrt{\rho}} \sqrt{\rho x^2 - (n - 1)^2}\right) - \frac{1}{\sqrt{\rho}} \cos\left(\frac{1 - \rho}{\sqrt{\rho}} \sqrt{\rho x^2 - n^2}\right)\right].$$

If $x^2 \gg n^2$ these expressions may be further simplified:

$$U_n(x, \rho x) = \frac{2}{\pi} \frac{1}{\sqrt{\rho x}} \sin (1 - \rho)x,$$

$$S_n(x, \rho x) = -\frac{2}{\pi} \frac{1}{\sqrt{\rho x}} \cos (1 - \rho)x.$$

Substituting these values in (36) we get:

$$\frac{W_n(x, \rho x)}{U_n(x, \rho x)} = \rho x \cot (1 - \rho)x - n,$$

$$\frac{V_n(x, \rho x)}{Z_n(x, \rho x)} = -\rho x \tan (1 - \rho)x \frac{1 - \frac{n}{x} \frac{1 - \rho}{\rho} \cot (1 - \rho)x}{1 - \frac{n}{x} \tan (1 - \rho)x}.$$

With the restrictions $(1 - \rho)x < \pi/2$ and $n \ll x$ we may write:

$$\begin{aligned} \frac{W_n(x, \rho x)}{U_n(x, \rho x)} &= \rho x \cot (1 - \rho)x, \\ \frac{V_n(x, \rho x)}{Z_n(x, \rho x)} &= -\rho x \tan (1 - \rho)x. \end{aligned} \tag{46}$$

With (46) the characteristic equation (1) is

$$n^2 \left[\frac{1}{x_1^2} - \frac{1}{x_2^2}\right]^2 - \rho^2 \frac{x_2^2 - x_1^2}{x_2^2 - \epsilon x_1^2} \left[\frac{1}{x_1} \frac{J_n'(\rho x_1)}{J_n(\rho x_1)} + \frac{\epsilon}{x_2} \cot \delta x_2\right] \cdot \left[\frac{1}{x_2} \frac{J_n'(\rho x_1)}{J_n(\rho x_1)} - \frac{1}{x_2} \tan \delta x_2\right] = 0.$$

Solving this equation for $\cot \delta x_2$ we get:

$$\cot \delta x_2 = \frac{F}{2} + \sqrt{1 + \frac{F^2}{4}}, \tag{47}$$

in which

$$F = \frac{x_1}{x_2} \frac{J_n(\rho x_1)}{J_n'(\rho x_1)} \left[1 + \frac{n^2}{\epsilon \rho^2} \frac{(x_1^2 - x_2^2)(x_2^2 - \epsilon x_1^2)}{x_1^4 x_2^2}\right] - \frac{x_2}{\epsilon x_1} \frac{J_n'(\rho x_1)}{J_n(\rho x_1)}.$$

To evaluate (47) we specify a value of $\Delta\beta/\beta_{nm}$ and enter the right hand side of (47) with

$$\begin{aligned} x_1^2 &= p_{nm}^2 - 2 \frac{\Delta\beta}{\beta_{nm}} \beta_{nm}^2 a^2 \left(1 + \frac{1}{2} \frac{\Delta\beta}{\beta_{nm}}\right), \\ x_2^2 &= (\epsilon - 1)\beta_{nm}^2 a^2 + x_1^2, \\ \rho &= 1 - \delta_1 \end{aligned}$$

in which δ_1 is a first order approximation as given by (10) for a particular mode. Equation (47) then yields a value δ_2 which is usually accurate enough. To improve the accuracy the same calculation is repeated using δ_2. Since for small values of δ a change in δ affects the right hand side only slightly this method converges rapidly.

APPENDIX II

The TE_{01} *Attenuation in the Dielectric-Coated Waveguide*

The attenuation constant of a transmission line can be expressed as

$$\alpha = \frac{1}{2}\frac{P_M}{P}$$

in which P_M is the power dissipated per unit length in the line and P is the total transmitted power. For the dielectric-coated guide the power P_M is dissipated in the metal walls with finite conductivity σ.

For TE_{om} waves the wall currents are $i = H_z(a)$ (H_z axial component of the magnetic field) and therefore

$$P_M = \frac{1}{2}\int_0^{2\pi} \frac{ii^*}{t\cdot\sigma}\, a\, d\varphi$$

$$= \frac{\pi a}{t\sigma} H_z(a) H_z^*(a)$$

where t = skin depth and σ = conductivity.

The total transmitted power is

$$P = -\frac{1}{2}\int_0^{2\pi}\int_0^a E_\varphi H_r^* r dr\, d\varphi.$$

We introduce expressions for the field components, which are listed elsewhere[3, 4] and carry out the integration. Finally we express the wall current attenuation in terms of the functions (2) and (35);

$$\alpha_M = \frac{\alpha_{\epsilon m}}{p_{om}^2} x_2^2 \sqrt{\frac{x_2^2 - x_1^2}{x_2^2 - \epsilon x_1^2}} \left\{ 1 + \frac{\pi^2}{2} \rho x_2 \cdot \left(\frac{x_2^2}{x_1^2} - 1\right)\left(U_1(x_2, \rho x_2) + \frac{\rho x_2}{2} R_1(x_2, \rho x_2)\right) R_1(x_2, \rho x_2) \right\}^{-1} \tag{48}$$

To get an approximation for a thin dielectric coat we use the expressions (37) and (38) and obtain

$$\frac{\Delta\alpha_M}{\alpha_{om}} = \frac{\alpha_M - \alpha_{\epsilon m}}{\alpha_{om}} = (\epsilon - 1)\frac{p^2_{om}}{\nu^2_{om}}\delta^2. \tag{49}$$

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